

STRICHARTZ TYPE ESTIMATES AND THE WELL POSEDNESS OF AN ENERGY CRITICAL 2D WAVE EQUATION IN A BOUNDED DOMAIN

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ABSTRACT. We study the well-posedness of the Cauchy problem with Dirichlet or Neumann boundary conditions associated to an H^1 -critical semilinear wave equation on a smooth bounded domain $\Omega \subset \mathbb{R}^2$. First, we prove an appropriate Strichartz type estimate using the L^q spectral projector estimates of the Laplace operator. Our proof follows Burq-Lebeau-Planchon [5]. Then, we show the global well-posedness when the energy is below or at the threshold given by the sharp Moser-Trudinger inequality. Finally, in the *supercritical* case, we prove an instability result using the finite speed of propagation and a quantitative study of the associated ODE with oscillatory data.

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1. INTRODUCTION

Recall the following semi-linear wave equation

$$(1.1) \quad \begin{aligned} (\partial_t^2 - \Delta)u + f(u) &= 0 \quad \text{in } \mathbb{R}_t \times \Omega_x, \\ u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x), \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a smooth bounded domain, Δ denotes the Laplace-Beltrami operator acting on the space variable x , and the nonlinearity f is an odd function satisfying $f(0) = 0$ and $uf(u) \geq 0$. The unknown $u = u(t, x)$ is a real-valued function. Note that the above assumptions on f include the massive case, namely the Klein-Gordon equation.

The most studied nonlinear model is when $f(u) = |u|^{p-1}u$, with $p > 1$. In the case of the whole space $\Omega = \mathbb{R}^d$ and $d \geq 3$, there is a large literature on the local and global solvability of (1.1) in the scale of the Sobolev spaces H^s i.e. the initial data $(u_0, u_1) \in H^s \times H^{s-1}$. Among others, we refer the interested readers to, [9, 11, 13, 18, 20, 30, 31, 32, 33, 38].

For the global solvability in the energy space $(u_0, u_1) \in H^1 \times L^2$, there are mainly three cases. In the subcritical case where $p < p_c = 1 + \frac{4}{d-2}$, Ginibre and Velo [11] have shown that problem (1.1) has a unique solution in the space $C(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$. In the critical case, $p = p_c$, the first global well-posedness result was obtained by Struwe in the radial case [38]. Then Grillakis in [13] established the existence of global smooth solutions for smooth data when $d = 3$. For higher dimensions, Shatah-Struwe [32, 33] proved the global solvability for data in the energy space. The quintic Klein-Gordon equation in 3D was globally solved by Kapitanski [21]. In the supercritical case, $p > p_c$, the local well-posedness was recently solved by Kenig-Merle [23] but for initial data in the homogeneous Sobolev spaces $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ with $1 < s_p < 3/2$. In the energy space this is still an open problem except for some partial results about some kind of "illposedness". See [25, 26, 7] for more details.

If Ω is the complement of a strictly convex, smooth and compact obstacle \mathcal{O} , problem (1.1) with a Dirichlet boundary condition $u|_{\partial\Omega} = 0$ was solved by Smith and Sogge for the 3D quintic equation. See [35]. The case of a smooth bounded domain in \mathbb{R}^3 was recently solved by Burq-Lebeau-Planchon [5], and Burq-Planchon [4] who showed the existence and uniqueness of a global solution for data in the energy space. The major difficulty in proving such a result is to establish Strichartz type estimates. Let us recall a few historical facts about these estimates.

For a manifold Ω of dimension $d \geq 2$ equipped with a Riemannian metric g , Strichartz estimates are a family of space time integral estimates on solutions : $u(t, x) : (-T, T) \times \Omega \longrightarrow \mathbb{R}$ to the wave equation

$$\partial_t^2 u - \Delta_g u = 0 \quad \text{in } (-T, T) \times \Omega_x$$

$$u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x).$$

Local Strichartz estimates state that

$$(1.2) \quad \|u\|_{L^q((-T, T), L^r(\Omega))} \leq C_T (\|u_0\|_{H^s(\Omega)} + \|u_1\|_{H^{s-1}(\Omega)}),$$

where $H^s(\Omega)$ denotes the L^2 -based Sobolev space, $2 \leq q \leq \infty$ and $2 \leq r < \infty$ satisfy

$$(1.3) \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}.$$

Estimates (1.2) are said global if the constant C_T is T -independent. Estimates involving $q = \infty$ hold when $(n, q, r) \neq (3, 2, \infty)$, but typically require the use of Besov spaces.

If $\Omega = \mathbb{R}^d$ and $g_{ij} = \delta_{ij}$, R. Strichartz proved in [37] global estimates for the wave and Shrödinger

equations in the diagonal case i.e. $q = r$. Then, Ginibre-Velo [12] and Lindblad-Sogge [27] generalized them to the other cases, see also Kato [22] and Cazenave-Weissler [6].

For general manifolds, phenomena such as the existence of trapped geodesics or the finiteness of the volume can preclude the development of global estimates, leading us to consider just local in time estimates.

In the case of a compact manifold without boundary, using the finite speed of propagation and working in coordinate charts, the problem is reduced to the proof of the local Strichartz estimates for the variable coefficients wave operators on \mathbb{R}^d . In this context, Kapitanski in [19] and Mockenhaupt-Seeger-Sogge in [28] established such inequalities for operators with smooth coefficients. Smith in [34] and Tataru in [39] have shown Strichartz estimates for operators with $C^{1,1}$ coefficients. For more details, see [3].

If Ω is a manifold with strictly geodesically-concave boundary, Smith-Sogge (see[35]) have shown Strichartz estimates for a larger range of exponents in (1.3).

Using the $L^r(\Omega)$ estimates for the spectral projector obtained by Smith-Sogge [36], Burq-Lebeau-Planchon established Strichartz estimates for a bounded domain for a certain range of triples (q, r, s) , see [5]. Recently, Blair-Smith-Sogge in [3] expanded the range of indices q and r obtained in [5] and also to other dimensions.

In the case where Ω is a compact convex domain in \mathbb{R}^2 , Ivanovici has very recently shown in [17] that (1.2) cannot hold when $r > 4$ if $2/q + 1/r = 1/2$.

Going back to the well-posedness issues, observe that in 2D all nonlinearities f with polynomial growths are “subcritical” for the H^1 norm. This is due to the limit case of the Sobolev embedding. So, the choice of an exponential nonlinearity appears to be quite natural. Such nonlinearity was investigated by Nakamura and Ozawa [30, 31]. They showed the global solvability and established the asymptotic in time when the initial data is sufficiently small. In a recent work, Ibrahim-Majdoub-Masmoudi [14] considered the case where $f(u) = ue^{4\pi u^2}$. They have quantified the size of the initial data for which one has global well-posedness. More precisely, let

$$E_0 = \|u_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} \frac{e^{4\pi u_0^2} - 1}{4\pi} dx.$$

Then, solutions with $E_0 \leq 1$ exist for all time. However, in the “supercritical” case i.e. $E_0 > 1$, the same authors have shown an instability result (see [16]), by proving the non uniform continuity of the solution map. Recently, a similar trichotomy was also established by Colliander-Ibrahim-Majdoub-Masmoudi for the nonlinear Schrödinger equation with the same type of nonlinearity. See [8].

In this paper, we propose to extend the above results to the case of bounded 2D domains. We establish a trichotomy in the dynamic for both Dirichlet and Neumann type boundary conditions. More precisely, consider the 2D, H^1 -critical wave equation

$$(1.4) \quad \begin{aligned} &(\partial_t^2 - \Delta_D)u + u(e^{4\pi u^2} - 1) = 0 \quad \text{in } \mathbb{R}_t \times \Omega_x \\ &u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x) \end{aligned}$$

$$u|_{\mathbb{R}_t \times \partial\Omega_x} = 0,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $u = u(t, x)$ is a real-valued function and Δ_D denotes the Laplace-Beltrami operator with Dirichlet boundary conditions. The initial data (u_0, u_1) are in the energy space $H_0^1(\Omega) \times L^2(\Omega)$.

A solution $u \in \mathcal{C}([0, T], H^1) \cap \mathcal{C}^1([0, T], L^2)$ of the Cauchy Problem (1.4) satisfies the following conservation law

$$(1.5) \quad E(u, t) = \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{4\pi} dx = E(u, 0).$$

A priori, one can estimate the nonlinear part of the energy using the following sharp Moser-Trudinger-type inequality, see for example [29], [40].

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $\alpha \leq 4\pi$. There exists a constant $C(\Omega) > 0$ such that*

$$(1.6) \quad \sup_{\|\nabla u\|_{L^2(\Omega)} \leq 1} \int_{\Omega} e^{\alpha u^2} dx = C(\Omega) < +\infty.$$

Moreover, this inequality is sharp in the sense that for any $\alpha > 4\pi$, the supremum in (1.6) is infinite.

In our paper we take $\alpha = 4\pi$, and then the discussion will be based on the size of the initial data in the energy space. More precisely, we distinguish the cases $E_0 \leq 1$ and $E_0 > 1$ where $E_0 = E(u, 0)$ is the energy of a solution u .

Our first result is the following Strichartz type estimate¹.

Theorem 1.2. *Suppose that $u \in \mathcal{C}([0, T], H_0^1) \cap \mathcal{C}^1([0, T], L^2)$ solves the linear inhomogeneous linear wave equation with Dirichlet boundary condition and $f \in L^1([0, T], L^2)$*

$$(\partial_t^2 - \Delta_D)u = f \quad \text{in } \mathbb{R}_t \times \Omega_x$$

$$(1.7) \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x)$$

$$u|_{(0, T) \times \partial\Omega_x} = 0.$$

Then, a constant C_T exists such that

$$(1.8) \quad \|u\|_{L^8((0, T), C^{1/8}(\Omega))} \leq C_T (\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^1((0, T), L^2(\Omega))}).$$

To prove this estimate, we follow the same approach of Burq-Lebeau-Planchon [5] in the case of a bounded domain $\Omega \subset \mathbb{R}^3$. Their idea is based on a recent result established by Smith and Sogge [36] to control the $L^5 W_{10}^{3,5}$ norm of the solution of the free wave equation by the energy norm.

To estimate the $L_T^1 L_x^2$ norm of the nonlinear term $u(e^{4\pi u^2} - 1)$, we remark that its $L^2(\Omega)$ norm already doubles the exponent 4π . Therefore, the inequality (1.6) is insufficient to control it. To overcome this difficulty, we use the following logarithmic inequality with sharp constant proved in [15].

Proposition 1.3. *For any real number $\lambda > \frac{4}{\pi}$ there exists a constant C_λ such that, for any function u belonging to $H_0^1(\Omega) \cap \dot{C}^{1/8}(\Omega)$, we have*

$$(1.9) \quad \|u\|_{L^\infty}^2 \leq \lambda \|\nabla u\|_{L^2(\Omega)}^2 \log \left(C_\lambda + \frac{\|u\|_{\dot{C}^{1/8}(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}} \right).$$

Moreover, the above inequality does not hold for $\lambda = \frac{4}{\pi}$.

¹This result was stated and proved in early 2009, and now it is embedded in Theorem 1.1 in [3]. However its proof is different.

Recall that for $0 < \alpha < 1$, \dot{C}^α denotes the homogeneous Hölder space: the set of continuous functions u whose norm $\|u\|_{\dot{C}^\alpha} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$ is finite. The inhomogeneous Hölder space is $C^\alpha = \dot{C}^\alpha \cap L^\infty$ endowed with the norm $\|u\|_{C^\alpha} = \|u\|_{\dot{C}^\alpha} + \|u\|_{L^\infty}$.

Using the above propositions we can show, through a fixed point argument, the existence of local in time solutions given by the following result.

Theorem 1.4. *Assume that $\|\nabla u_0\|_{L^2(\Omega)} < 1$. Then, there exists a time $T > 0$ and a unique solution u to problem (1.4), $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$. Moreover, $u \in L^8([0, T], C^{1/8}(\Omega))$ and satisfies the energy conservation, for all $0 \leq t < T$.*

Based on the above result and the sharp Moser-Trudinger inequality, we propose as in [14] the following definition.

Definition 1.5. *Let $E_0 = E(u, t = 0)$ given by (1.5). The Cauchy problem (1.4) is said to be*

- *Subcritical if $E_0 < 1$.*
- *Critical if $E_0 = 1$*
- *Supercritical if $E_0 > 1$.*

Thanks to the energy identity (1.5) and the local existence result, we can easily show the global existence in the subcritical case as stated in the following Theorem.

Theorem 1.6. *For any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with energy $E_0 < 1$ there is a unique global solution $u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$. Moreover, this solution $u \in L_{loc}^8(\mathbb{R}, C^{1/8}(\Omega))$ and satisfies (1.5).*

In the critical case we cannot apply the same arguments used in the subcritical case. This is due to the fact that the conservation of the energy only does not rule out the possibility for the solution to (at least formally) concentrate in the sense that

$$\limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\Omega)} = 1.$$

In such a case, we emphasize on the fact that we do not know any nonlinear estimate. Therefore, we use a multiplier techniques, we show that such concentration phenomena cannot occur and thus solutions are indeed global.

Theorem 1.7. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with energy $E_0 = 1$. There is a unique global solution u in the space $C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$ with the initial data (u_0, u_1) . Moreover, this solution $u \in L_{loc}^8(\mathbb{R}, C^{1/8}(\Omega))$ and satisfies (1.5).*

In the supercritical case, we shall prove that problem (1.4) is ill-posed. Precisely, we prove

Theorem 1.8. *Let $0 < \eta < 1$ be small enough. There exist a sequence of positive real number (t_k) tending to zero and two sequences (v_k^η) and (w_k^η) of solutions of (1.4) satisfying, for any $\varepsilon > 0$*

$$(1.10) \quad \|(v_k^\eta - w_k^\eta)(t = 0, \cdot)\|_{H_0^1(\Omega)}^2 + \|\partial_t(v_k^\eta - w_k^\eta)(t = 0, \cdot)\|_{L^2(\Omega)}^2 \leq \varepsilon$$

and,

$$(1.11) \quad 0 < E(w_k^\eta, 0) - 1 \leq \eta^2 \quad \text{and} \quad 0 < E(v_k^\eta, 0) - 1 \leq 3\eta^2 e^3$$

when k is large enough. Moreover,

$$(1.12) \quad \liminf_{k \rightarrow \infty} \|\partial_t(v_k^\eta - w_k^\eta)(t_k, \cdot)\|_{L^2(\Omega)}^2 \geq C.$$

The constant C depends only on η .

To prove this Theorem, we proceed in a similar way as in [16]. Their idea is based on the approximation of the solution of the PDE by the solution of its corresponding ODE (without the “diffusion term”). The special choice of the concentrating data combined to the finite speed of propagation guarantee that the two solutions indeed coincide in a backward light cone. Then a “decoherence” type phenomena is shown for the ODE regime given the periodicity of its solutions. The local character of the proof of [16] enables us to adapt it in our setting. This strategy was originally initiated by Kuksin [24] and developed by Christ-Colliander-Tao [7].

Remark 1.9. *The results of this paper remain true if we replace the Dirichlet by Neumann boundary conditions. This is due to the fact that we use only u and $\partial_t u$ as test functions. This considerably simplifies our proof compared to the 3D quintic problem, where in addition $x \cdot \nabla u$ is used. That term gave rise of new boundary terms which needed more care to control. We refer to [5] and [4] for full details. Also, thanks to Poincaré inequality, our results hold in the massive case i.e. the Klein-Gordon equation.*

This paper is organized as follows: in the next section, we introduce the notation used throughout this paper. Section two is devoted to the complete proof of our Strichartz estimates. In section three, we combine the latter estimates with the energy identity and the sharp logarithmic inequality to establish, through a standard fixed point argument, the local existence results. In section four, we focus on the proofs of Theorem 1.6 and Theorem 1.7. In the last section we prove the instability result given by Theorem 1.8.

2. NOTATION

For $s \geq 0$, let $H_D^s(\Omega)$ be the domain of $(-\Delta_D)^{s/2}$ and $H_0^s(\Omega)$ be the closure in $H^s(\Omega)$ of the set of smooth and compactly supported functions. Note that the space $H_D^s(\Omega)$ coincides with $H_0^s(\Omega)$ for $0 \leq s < \frac{3}{2}$, and that when $s = 1$, $H_0^1(\Omega)$ equipped with the inner product of $H^1(\Omega)$ is a Hilbert space. In this paper, the space $H_0^1(\Omega)$ will be endowed with the Dirichlet norm $\|u\|_{H_D^1(\Omega)}^2 = \int_{\Omega} |\nabla u(x)|^2 dx$.

It is well known that in our setting, the operator $-\Delta_D$ has a complete set of eigenvalues $\{\lambda_j^2\}_{j=0}^{\infty}$ and eigenfunctions. Let $m(\lambda_j)$ denote the multiplicity of λ_j^2 , and $e_{\lambda_{j,k}}$ be the k^{th} eigenvector in

the eigenspace corresponding to the eigenvalue λ_j^2 . Then define $\Pi_{\lambda_j} u = \sum_{k=1}^{m(\lambda_j)} \langle u, e_{\lambda_{j,k}} \rangle e_{\lambda_{j,k}}$,

where \langle, \rangle stands for the L^2 inner product. For any $\lambda > 1$, denote by χ_{λ} the spectral projection given by

$$\chi_{\lambda} u = \sum_{\{j / \lambda \leq \lambda_j < \lambda+1\}} \Pi_{\lambda_j} u.$$

Finally, let $|D| := \sqrt{-\Delta_D}$. For any $0 < S < T$ and $x_0 \in \Omega$, define :

$$K_S^T(x_0) = \{(x, t) / |x - x_0| < t, S < t < T, x \in \Omega\}, \quad \text{the backward light cone}$$

$$(2.13) \quad M_S^T(x_0) = \{(x, t) / |x - x_0| = t, x \in \Omega, S < t < T\} \quad \text{its mantle}$$

and for fixed t

$$D_t(x_0) = \{x / |x - x_0| < t\} \cap \Omega \quad \text{its space like sections.}$$

Observe that

$$\partial K_S^T(x_0) = ([S, T] \times \partial\Omega) \cap K_S^T(x_0) \cup D_S \cup D_T \cup M_S^T(x_0).$$

Finally, let $e(u)$ be the energy density

$$(2.14) \quad e(u) = (\partial_t u)^2 + |\nabla u|^2 + \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{4\pi}.$$

When $x_0 = 0$, we remove the dependence upon x_0 in the above notation. Let \mathcal{E}_T be the space defined as follows

$$(2.15) \quad \mathcal{E}_T = C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap L^8([0, T], C^{1/8}(\Omega)).$$

Set

$$(2.16) \quad \|u\|_T = \sup_{t \in [0, T]} (\|u(t, \cdot)\|_{H_0^1(\Omega)} + \|\partial_t u(t, \cdot)\|_{L^2(\Omega)}) + \|u\|_{L^8([0, T], C^{1/8}(\Omega))}.$$

Recall that \mathcal{E}_T equipped with the norm $\|\cdot\|_T$ is a complete space.

3. STRICHARTZ ESTIMATE

In this section, we prove our appropriate Strichartz estimate given by Theorem 1.2. The proof follows Burq-Lebeau-Planchon [5]. It is based on an estimate in Lebesgue spaces of the spectral projector χ_λ . This estimate is due to Smith-Sogge [36]. First, we recall this estimate in two space dimensions.

Proposition 3.1. [Smith-Sogge [36]] *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then the following estimate*

$$(3.17) \quad \|\chi_\lambda u\|_{L^q(\Omega)} \leq C \lambda^{\frac{2}{3}(\frac{1}{2} - \frac{1}{q})} \|u\|_{L^2(\Omega)}$$

holds for $2 \leq q \leq 8$.

Proof of Theorem 1.2. In this proof, we distinguish two cases.

First case : Estimate for the homogeneous problem i.e. when $f = 0$.

In this case, Duhamel's formula gives

$$u = \cos(t|D|)u_0 + \frac{\sin(t|D|)}{|D|}u_1$$

where

$$\cos(t|D|) := \left(\frac{e^{it|D|} + e^{-it|D|}}{2} \right), \quad \sin(t|D|) := \left(\frac{e^{it|D|} - e^{-it|D|}}{2i} \right)$$

and $\mathcal{L}(t)u_0 := e^{\pm it|D|}u_0$ is the solution u of $\partial_t u = \pm i|D|u$ and $u(t=0) = u_0$. By Minkowski inequality

$$\|u\|_{L^8((0,1), C^{1/8}(\Omega))} \leq \|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), C^{1/8}(\Omega))} + \|\mathcal{L}(\cdot)\left(\frac{u_1}{|D|}\right)\|_{L^8((0,1), C^{1/8}(\Omega))}.$$

Therefore, we need to estimate $\|\mathcal{L}(\cdot)u_0\|$ in $L^8((0,1), C^{1/8}(\Omega))$.

Step 1 : We show that

$$(3.18) \quad \|e^{itA}u_0\|_{L^8((0,2\pi), L^8(\Omega))} \leq C \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)},$$

where A is the “modified” $|D|$ operator with integer eigenvalues i.e.

$$A(e_\lambda) = [\lambda]e_\lambda.$$

The notation $[\cdot]$ stands for the integer part and e_λ is an eigenfunction of $-\Delta_D$ associated to the eigenvalue λ^2 (Hence an eigenfunction of $|D|$ associated to the eigenvalue λ).

Since u_0 is in $L^2(\Omega)$, we can write

$$u_0(x) = \sum_{\lambda \in \sigma(\sqrt{-\Delta_D})} \langle u_0, e_\lambda \rangle e_\lambda(x) =: \sum_{\lambda \in \sigma(\sqrt{-\Delta_D})} u_\lambda e_\lambda(x),$$

where $\sigma(\sqrt{-\Delta_D})$ denotes the spectrum of $\sqrt{-\Delta_D}$.
So,

$$\begin{aligned} e^{itA}u_0(x) &= \sum_{\lambda \in \sigma(\sqrt{-\Delta_D})} e^{itA}u_\lambda e_\lambda(x) \\ &= \sum_{\lambda \in \sigma(\sqrt{-\Delta_D})} e^{it[\lambda]}u_\lambda e_\lambda(x). \end{aligned}$$

Setting $k = [\lambda]$, we have

$$e^{itA}u_0(x) = \sum_{k=1}^{\infty} e^{itk}C_k(x),$$

where the Fourier coefficient $C_k(x)$ is given by

$$C_k(x) = \sum_{\substack{\lambda \in \sigma(\sqrt{-\Delta_D}) \\ k \leq \lambda < k+1}} u_\lambda e_\lambda(x).$$

Thanks to the 1D Sobolev embedding, $H^{\frac{1}{2}-\frac{1}{q}}(0, 2\pi) \hookrightarrow L^q(0, 2\pi)$ for all $q \geq 2$, we have

$$\begin{aligned} \|e^{itA}u_0\|_{L^8(\Omega, L^8(0, 2\pi))}^2 &\lesssim \left(\int_{\Omega} \|e^{itA}u_0(x)\|_{H^{\frac{3}{8}}(0, 2\pi)}^8 dx \right)^{1/4} \\ &\lesssim \left\| \|e^{itA}u_0(x)\|_{H^{\frac{3}{8}}(0, 2\pi)}^2 \right\|_{L^4(\Omega)}. \end{aligned}$$

Then, Parseval's formula gives

$$\begin{aligned} \|e^{itA}u_0(x)\|_{H^{\frac{3}{8}}(0, 2\pi)}^2 &= \sum_{k \geq 1} (1+k)^{3/4} \|e^{itk}C_k(x)\|_{L^2(0, 2\pi)}^2 \\ &\lesssim \sum_{k \geq 1} (1+k)^{3/4} |C_k(x)|^2. \end{aligned}$$

Now applying Minkowski inequality and using estimate (3.17), we obtain

$$\begin{aligned}
\|e^{itA}u_0\|_{L^8(\Omega, L^8(0, 2\pi))}^2 &\lesssim \left\| \sum_{k \geq 1} (1+k)^{3/4} |C_k(x)|^2 \right\|_{L^4(\Omega)} \\
&\lesssim \sum_{k \geq 1} (1+k)^{3/4} \|C_k\|_{L^8(\Omega)}^2 = C \sum_{k \geq 1} (1+k)^{3/4} \|\chi_k u_0\|_{L^8(\Omega)}^2 \\
&\lesssim \sum_{k \geq 1} (1+k)^{\frac{3}{4}} k^{\frac{1}{2}} \|\chi_k u_0\|_{L^2(\Omega)}^2 \\
&\lesssim \sum_{k \geq 1} (1+k)^{\frac{5}{4}} \sum_{\substack{\lambda \in \sigma(\sqrt{-\Delta_D}) \\ k \leq \lambda < k+1}} |u_\lambda|^2 \sim \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}^2,
\end{aligned}$$

which gives

$$\|e^{itA}u_0\|_{L^8(\Omega, L^8(0, 2\pi))}^2 \lesssim \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}^2$$

as desired.

Step 2 : We prove (3.18) for the operator $\mathcal{L}(\cdot)$.

$$(3.19) \quad \|\mathcal{L}(\cdot)u_0\|_{L^8((0, 2\pi), L^8(\Omega))} \leq C \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}.$$

Let $v = e^{it|D|}u_0$. It is clear that v satisfies

$$\begin{cases} (\partial_t - iA)v = (-iA + i|D|)v \\ v|_{t=0} = u_0, \end{cases}$$

and according to Duhamel's formula

$$v(t, x) = e^{itA}u_0(x) + \int_0^t e^{i(t-s)A}(-iA + i|D|)v(s, x) ds.$$

So, using Hölder in the second estimate

$$\begin{aligned}
\|v(\cdot, x)\|_{L^8(0, 2\pi)} &\leq \|e^{itA}u_0(x)\|_{L^8(0, 2\pi)} + \left\| \int_0^t e^{i(t-s)A}(-iA + i|D|)v(s, x) ds \right\|_{L^8(0, 2\pi)} \\
&\lesssim \|e^{itA}u_0(x)\|_{L^8(0, 2\pi)} + \left(\int_0^{2\pi} \int_0^{2\pi} |e^{i(t-s)A}(-iA + i|D|)v(s, x)|^8 ds dt \right)^{1/8} \\
&\lesssim (\|e^{itA}u_0(x)\|_{L^8(0, 2\pi)} + \left(\int_0^{2\pi} \|e^{i(t-s)A}(-iA + i|D|)v(s, x)\|_{L^8(0, 2\pi)}^8 ds \right)^{1/8}).
\end{aligned}$$

Applying (3.18)

$$\begin{aligned}
\|v\|_{L^8(\Omega, L^8(0, 2\pi))} &\lesssim \left(\|u_0\|_{H_D^{\frac{5}{8}}(\Omega)} + \left(\int_\Omega \int_0^{2\pi} \|e^{i(t-s)A}(-iA + i|D|)v(s, x)\|_{L^8(0, 2\pi)}^8 ds dx \right)^{1/8} \right) \\
&\lesssim \left(\|u_0\|_{H_D^{\frac{5}{8}}(\Omega)} + \left(\int_0^{2\pi} \|e^{i(t-s)A}(-iA + i|D|)v(s)\|_{L^8(\Omega, L^8(0, 2\pi))}^8 ds \right)^{1/8} \right).
\end{aligned}$$

Since the operator $A - |D|$ is bounded on $H_D^{\frac{5}{8}}$, then

$$\begin{aligned} \|v\|_{L^8(\Omega, L^8(0, 2\pi))} &\lesssim \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)} + \left(\int_0^{2\pi} \|(A - |D|)v(s)\|_{H_D^{\frac{5}{8}}(\Omega)}^8 ds \right)^{1/8} \\ &\lesssim \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)} + \left(\int_0^{2\pi} \|v(s)\|_{H_D^{\frac{5}{8}}(\Omega)}^8 ds \right)^{1/8} \\ &\lesssim \|u_0\|_{H_D^{\frac{5}{8}}} + C \left(\int_0^{2\pi} \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}^8 ds \right)^{1/8} \\ &\lesssim \|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}, \end{aligned}$$

where we used $\sup_{t \in [0, 2\pi]} \|v(t)\|_{H_D^{\frac{5}{8}}(\Omega)} = \|e^{it|D|}u_0\|_{H_D^{\frac{5}{8}}(\Omega)} \leq C\|u_0\|_{H_D^{\frac{5}{8}}(\Omega)}$ in the last inequality.

As a consequence, we obtain (3.19) as desired.

Step 3 : We show that

$$\|u\|_{L^8((0,1), C^{1/8}(\Omega))} \lesssim (\|u_0\|_{H_D^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).$$

Recall the following elliptic regularity result:

$$-\Delta_D u + u = g \in L^q(\Omega) \text{ and } u|_{\partial\Omega} = 0 \Rightarrow u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \text{ and } \|u\|_{W^{2,q}(\Omega)} \lesssim \|g\|_{L^q(\Omega)}.$$

Assuming $u_0 \in C_0^\infty(\Omega)$, we have for almost all t

$$-\Delta_D \mathcal{L}(t)u_0 + \mathcal{L}(t)u_0 = \mathcal{L}(t)(-\Delta_D u_0 + u_0) \in L^8(\Omega).$$

Thus

$$\begin{aligned} \|\mathcal{L}(t)u_0\|_{W^{2,8}(\Omega)} &\lesssim \|-\Delta_D(\mathcal{L}(t)u_0) + \mathcal{L}(t)u_0\|_{L^8(\Omega)} \\ &\lesssim (\|\mathcal{L}(t)(\Delta_D u_0)\|_{L^8(\Omega)} + \|\mathcal{L}(t)u_0\|_{L^8(\Omega)}), \end{aligned}$$

and therefore

$$\|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), W^{2,8}(\Omega))} \lesssim (\|\mathcal{L}(\cdot)(\Delta_D u_0)\|_{L^8((0,1), L^8(\Omega))} + \|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), L^8(\Omega))}).$$

Applying (3.19) to $\Delta_D u_0$ we obtain

$$\|\mathcal{L}(\cdot)(\Delta_D u_0)\|_{L^8((0,1), L^8(\Omega))} \leq \|\mathcal{L}(\cdot)(\Delta_D u_0)\|_{L^8((0,2\pi), L^8(\Omega))} \lesssim \|\Delta_D u_0\|_{H_D^{\frac{5}{8}}(\Omega)} \lesssim \|u_0\|_{H_D^{\frac{21}{8}}(\Omega)}.$$

Consequently

$$(3.20) \quad \|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), W^{2,8}(\Omega))} \lesssim \|u_0\|_{H_D^{\frac{21}{8}}(\Omega)}.$$

Applying the complex interpolation to (3.19) and (3.20) with $\theta = \frac{3}{16}$, we have

$$(3.21) \quad \|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), W^{\frac{3}{8},8}(\Omega))} \lesssim \|u_0\|_{H_D^1(\Omega)}.$$

Now, by Sobolev embedding, we have for all $p \leq 8$

$$W^{\frac{1}{8} + \frac{2}{p}, 8}(\Omega) \hookrightarrow C^{\frac{2}{p} - \frac{1}{8}}(\Omega) \hookrightarrow C^{\frac{1}{8}}(\Omega).$$

Thus, we can rewrite (3.21) as

$$(3.22) \quad \|\mathcal{L}(\cdot)u_0\|_{L^8((0,1), C^{1/8}(\Omega))} \lesssim \|u_0\|_{H_D^1(\Omega)}$$

which implies that

$$\|u\|_{L^8((0,1),C^{1/8}(\Omega))} \lesssim (\|u_0\|_{H_D^1(\Omega)} + \|\frac{u_1}{|D|}\|_{H_D^1(\Omega)}).$$

Finally, we use the fact that $|D|^{-1}$ is an isometry from $L^2(\Omega)$ to $H_0^1(\Omega)$ to conclude that

$$\|u\|_{L^8((0,1),C^{1/8}(\Omega))} \lesssim (\|u_0\|_{H_D^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).$$

Second case: An arbitrary $f \in L^1(L^2)$.

Thanks to Duhamel's formula

$$u(t, x) = \cos(t|D|)u_0 + \sin(t|D|)|D|^{-1}u_1 + \int_0^t \sin((t-s)|D|)|D|^{-1}f(s) ds.$$

Applying the result of the first case, we obtain

$$\begin{aligned} \|u\|_{L^8((0,1),C^{1/8}(\Omega))} &\lesssim \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right) \\ &+ \int_0^t \|\sin((t-s)|D|)|D|^{-1}f(s)\|_{L^8((0,1),C^{1/8}(\Omega))} ds \\ &\lesssim \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right) + \int_0^1 \| |D|^{-1}f(s) \|_{H_0^1(\Omega)} ds \quad (\text{by (3.22)}) \\ &\lesssim \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \int_0^1 \|f(s)\|_{L^2(\Omega)} ds \right) \\ &\lesssim \left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^1((0,1),L^2(\Omega))} \right). \end{aligned}$$

This completes the proof of Theorem 1.2. ■

Now we show that in the supercritical case, it is impossible to estimate the nonlinear term in any dual Strichartz norm. Our result stands for the solutions to the free wave equation which is the first iteration in any iterative scheme for the nonlinear problem. We emphasize on the fact that the linear energy is less than one, and the nonlinear one is slightly bigger than one (supercritical). More precisely we have

Proposition 3.2. *For any $\delta > 0$, there exists a sequence (v_k) of solutions of the free wave equation such that we have*

$$(3.23) \quad \|\nabla v_k(0)\|_{L^2}^2 + \|v_k(0)\|_{L^2}^2 + \|\partial_t v_k(0)\|_{L^2}^2 < 1, \quad \text{and} \quad E(v_k, t=0) \leq 1 + \delta$$

for k large, while for any $T > 0$, any $p, q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{2}{q} \leq 2$

$$(3.24) \quad \|f(v_k)\|_{L^p((0,T),L^q(\Omega))} \geq C_\delta \sqrt{k}.$$

Proof. Without loss of generality, we may assume that $0 \in \Omega$. Let $\delta > 0$, and choose p, q such that

$$\frac{1}{p} + \frac{2}{q} \leq 2.$$

For any $k \geq 1$, let v_k be the solution of the free wave equation with data

$$v_k(0, x) = \left(1 - \frac{2a}{k}\right)f_k(ax) \quad \text{and} \quad \partial_t v_k(0, x) = 0,$$

where $a > 1$ to be chosen in the sequel. The functions f_k are defined by

$$(3.25) \quad f_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ -\frac{\log|x|}{\sqrt{k\pi}} & \text{if } e^{-k/2} \leq |x| \leq 1 \\ \sqrt{\frac{k}{4\pi}} & \text{if } |x| \leq e^{-k/2}, \end{cases}$$

and were introduced in [29] to show the optimality of the exponent 4π in Trudinger-Moser inequality.

Let $a_1 > 1$ be sufficiently large such that the ball $B(0, 1/a_1) \subset \Omega$. For $a > a_1$ we have,

$$\|\nabla v_k(0)\|_{L^2(\Omega)}^2 = \frac{2}{k} \left(1 - \frac{2a}{k}\right)^2 \int_{\frac{1}{a}e^{-k/2}}^{\frac{1}{a}} \frac{dr}{r} = \left(1 - \frac{2a}{k}\right)^2$$

and

$$\begin{aligned} \|v_k(0)\|_{L^2(\Omega)}^2 &= \left(1 - \frac{2a}{k}\right)^2 \left[\frac{2}{ka^2} \int_{e^{-k/2}}^1 r \log^2 r \, dr + \frac{k}{2} \int_0^{\frac{1}{a}e^{-k/2}} r \, dr \right] \\ &\leq \frac{1}{2ka^2} \left(1 - \frac{2a}{k}\right)^2. \end{aligned}$$

As $v_k(0, x)$ can be extended (by zero outside its support) as an $H^1(\mathbb{R}^2)$, then the Trudinger-Moser inequality:

$$\|\nabla \varphi\|_{L^2(\mathbb{R}^2)} < 1 \implies \int_{\mathbb{R}^2} (e^{4\pi|\varphi|^2} - 1) dx \lesssim \frac{\|\varphi\|_{L^2(\mathbb{R}^2)}^2}{1 - \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}^2}$$

shows that

$$\int_{\Omega} (e^{4\pi v_k^2(0,x)} - 1) dx \leq \int_{\mathbb{R}^2} (e^{4\pi v_k(0,x)^2} - 1) dx \leq \frac{C}{a^3},$$

for an absolute constant C . Therefore, we can choose $a_2 > a_1$ such that $\frac{C}{a_2^3} \leq \delta$. Thus, for $a \geq a_2$ and k large enough we have

$$\|v_k(0)\|_{L^2(\Omega)}^2 + \|\nabla v_k(0)\|_{L^2(\Omega)}^2 + \int_{\Omega} (e^{4\pi v_k^2(0)} - 1) dx \leq 1 + \delta,$$

and (3.23) follows. Next, by the finite speed of propagation, we know that

$$v_k(t, x) = \left(1 - \frac{2a}{k}\right) \sqrt{\frac{k}{4\pi}}$$

for any (t, x) in the backward light cone

$$K_0^k := \{(x, t) : 0 \leq t \leq \frac{e^{-k/2}}{a} \text{ and } |x| \leq \frac{e^{-k/2}}{a} - t\}.$$

Thus for k large enough (eventually with respect to a)

$$\begin{aligned} v_k(e^{4\pi v_k^2} - 1) &\geq \left(1 - \frac{2a}{k}\right) \sqrt{\frac{k}{4\pi}} \left(\exp\left(\left(1 - \frac{2a}{k}\right)^2 k\right) - 1\right) \\ &\geq C\sqrt{k}e^k. \end{aligned}$$

Now choosing k larger so that $e^{-k/2} \leq T$, we have the estimate

$$\begin{aligned} \|v_k(e^{4\pi v_k^2} - 1)\|_{L^p((0,T), L^q(\Omega))} &\geq \|v_k(e^{4\pi v_k^2} - 1)\|_{L^p((0, \frac{1}{a}e^{-\frac{k}{2}}), L^q(|x| \leq \frac{1}{a}e^{-\frac{k}{2}} - t))} \\ &\geq C\sqrt{k}e^k \left(\frac{e^{-k/2}}{a} \right)^{\frac{2}{q} + \frac{1}{p}} \\ &\geq C \frac{\sqrt{k}}{a^2}, \end{aligned}$$

where we used the fact that $\frac{2}{q} + \frac{1}{p} \leq 2$ in the third inequality. ■

4. THE LOCAL EXISTENCE

In this section, we prove Theorem 1.4. We start by giving two Lemmas. The first one provides the nonlinear estimate needed for the fixed point argument. The second one will be used to show the unconditional uniqueness result². The \mathbb{R}^2 -counter parts of these Lemmas can be found in [14].

Lemma 4.1. *Fix a time $T > 0$ and $0 < A < 1$, and denote by $f(u) = u(e^{4\pi u^2} - 1)$. There exists $0 < \gamma = \gamma(A) < 8$ such that if*

$$u_1, u_2 \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap L^8([0, T], C^{1/8}(\Omega))$$

and

$$\sup_{t \in [0, T]} \|\nabla u_i(t, \cdot)\|_{L^2(\Omega)} \leq A \quad i = 1, 2,$$

then

$$\|f(u_1) - f(u_2)\|_{L_T^1(L^2(\Omega))} \leq c\|u_1 - u_2\|_T \left(T + T^{1-\frac{\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right),$$

where the norm $\|\cdot\|_T$ is defined by (2.16).

Proof. Thanks to the mean value theorem we can write

$$u_1(e^{4\pi u_1^2} - 1) - u_2(e^{4\pi u_2^2} - 1) = u[(1 + 8\pi u_\theta^2)e^{4\pi u_\theta^2} - 1]$$

with $u = u_1 - u_2$ and $0 \leq \theta \leq 1$. Set $u_\theta = (1 - \theta)u_1 + \theta u_2$. We have

$$\sup_{t \in [0, T]} \|D|u_\theta(t, \cdot)\|_{L^2(\Omega)} \leq A.$$

So

$$\|f(u_1) - f(u_2)\|_{L_T^1(L^2(\Omega))} = \left\| u[(1 + 8\pi u_\theta^2)e^{4\pi u_\theta^2} - 1] \right\|_{L_T^1(L^2(\Omega))}.$$

On the other hand, observe that for any $a > 0$ and $\varepsilon > 0$,

$$(4.26) \quad (1 + 2a)e^a - 1 \leq 2\left(1 + \frac{1}{\varepsilon}\right)(e^{(1+\varepsilon)a} - 1).$$

Then, Hölder inequality together with Sobolev embedding and the above observation yield

²Uniqueness in the energy space and not in just a subspace of it.

$$\begin{aligned}
\left\| u[(1 + 8\pi u_\theta^2)e^{4\pi u_\theta^2} - 1] \right\|_{L^2(\Omega)}^2 &\leq C_\varepsilon \left\| u(e^{4\pi(1+\varepsilon)u_\theta^2} - 1) \right\|_{L^2(\Omega)}^2 \\
&\leq C_\varepsilon \|u(t)\|_{L^{2+\frac{2}{\zeta}}(\Omega)}^2 \left\| (e^{4\pi(1+\varepsilon)u_\theta^2} - 1)^2 \right\|_{L^{1+\zeta}(\Omega)} \\
&\leq C_\varepsilon \|u(t)\|_{H^1(\Omega)}^2 e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\Omega)}^2} \left\| e^{4\pi(1+\varepsilon)u_\theta^2} - 1 \right\|_{L^{1+\zeta}(\Omega)},
\end{aligned}$$

for any $\varepsilon > 0$. Moreover, since $\|u_\theta\|_{H_0^1(\Omega)}^2 \leq A^2$, then the Moser-Trudinger inequality (1.6) implies that

$$\int_{\Omega} (e^{4\pi(1+\varepsilon)u_\theta^2} - 1)^{1+\zeta} dx \leq \int_{\Omega} (e^{4\pi(1+\varepsilon)(1+\zeta)u_\theta^2} - 1) dx \leq C(\Omega, A),$$

provided that $\varepsilon > 0$ and $\zeta > 0$ are chosen such that $(1 + \varepsilon)(1 + \zeta)A^2 < 1$.

Thanks to the log estimate (1.9) for $\lambda > 4/\pi$ there is a constant $C_\lambda > 1$ such that

$$e^{4\pi(1+\varepsilon)\|u_\theta(t)\|_{L^\infty(\Omega)}^2} \leq \exp \left(4\pi\lambda(1+\varepsilon) \| |D|u_\theta(t) \|_{L^2(\Omega)}^2 \log \left(C_\lambda + \frac{\|u_\theta\|_{C^{1/8}(\Omega)}}{\| |D|u_\theta(t) \|_{L^2(\Omega)}} \right) \right).$$

Using the fact that for any $B_1 > 1$, $B_2 > 0$, the function $x \mapsto x^2 \log(B_1 + \frac{B_2}{x})$ is non-decreasing, we deduce that

$$\begin{aligned}
e^{4\pi(1+\varepsilon)\|u_\theta(t)\|_{L^\infty(\Omega)}^2} &\leq \exp \left(4\pi(1+\varepsilon)\lambda A^2 \log \left(C_\lambda + \frac{\|u_\theta(t)\|_{C^{1/8}(\Omega)}}{A} \right) \right) \\
&\leq \left(C_\lambda + \frac{\|u_\theta(t)\|_{C^{1/8}(\Omega)}}{A} \right)^{4\pi(1+\varepsilon)\lambda A^2}.
\end{aligned}$$

Setting $\gamma = 2\pi\lambda(1+\varepsilon)A^2$, we have

$$\left\| u[(1 + 8\pi u_\theta^2)e^{4\pi u_\theta^2} - 1] \right\|_{L^2(\Omega)} \leq C_{(A,\Omega)} \|u(t)\|_{H^1(\Omega)} \left(C_\lambda + \frac{\|u_\theta(t)\|_{C^{1/8}(\Omega)}}{A} \right)^\gamma.$$

Now since $A < 1$, we can choose λ such that $0 < \gamma < 8$. Thus

$$\begin{aligned}
\int_0^T \left\| u[(1 + 8\pi u_\theta^2)e^{4\pi u_\theta^2} - 1] \right\|_{L^2(\Omega)} dt &\leq C_{(\Omega,A)} \|u\|_T \int_0^T \left(C + \frac{\|u_\theta(t)\|_{C^{1/8}(\Omega)}}{A} \right)^\gamma dt \\
&\leq C_{(\Omega,A)} \|u\|_T \left[T + \left\| \left(\frac{\|u_1(t)\|_{C^{1/8}(\Omega)}}{A} \right)^\gamma \right\|_{L_T^1} + \left\| \left(\frac{\|u_2(t)\|_{C^{1/8}(\Omega)}}{A} \right)^\gamma \right\|_{L_T^1} \right] \\
&\leq C_{(\Omega,A)} \|u\|_T \left[T + T^{\frac{8-\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right].
\end{aligned}$$

Finally, we obtain

$$\|f(u_1) - f(u_2)\|_{L_T^1(L^2(\Omega))} \leq C_{(\Omega,A)} \|u\|_T \left[T + T^{1-\frac{\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right]$$

as desired. ■

Lemma 4.2. *Let $F(u) = e^{4\pi u^2} - 1$ and $u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ be the solution of (1.4) with $u(t=0) = u_0$ such that $\|\nabla u_0\|_{L^2(\Omega)} < 1$. Then there exists a continuous real valued function $C(t)$, vanishing at zero such that*

$$\|F(u)\|_{L_T^1(L^2(\Omega))} \leq C(T).$$

Proof. Write the solution u of (1.4) as $u = v_L + \tilde{u}$ where v_L solves the free wave equation with the same data as u and \tilde{u} belongs to $C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ and solves the perturbed problem

$$(4.27) \quad \begin{cases} \square \tilde{u} = -(v_L + \tilde{u})(e^{4\pi(v_L + \tilde{u})^2} - 1) \\ \tilde{u}(0, x) = 0 \\ \partial_t \tilde{u}(0, x) = 0. \end{cases}$$

Recall the following trivial observations

$$(4.28) \quad \text{For all } \varepsilon > 0, \quad (\tilde{u} + v_L)^2 \leq (1 + \frac{1}{2\varepsilon})\tilde{u}^2 + (1 + 2\varepsilon)v_L^2$$

$$(4.29) \quad e^{x+y} - 1 = (e^x - 1)(e^y - 1) + (e^x - 1) + (e^y - 1)$$

and

$$(4.30) \quad \text{for all } x \geq 0, \text{ and } \alpha > 1, \quad (e^x - 1)^\alpha \leq e^{\alpha x} - 1.$$

Set $a = (1 + 2\varepsilon)$ and $b = (1 + \frac{1}{2\varepsilon})$, then observations (4.28) and (4.29) imply

$$\begin{aligned} \left(e^{4\pi(v_L + \tilde{u})^2} - 1\right)^2 &\leq \left(e^{4\pi(av_L^2 + b\tilde{u}^2)} - 1\right)^2 \\ &\leq 3 \left[(e^{4\pi av_L^2} - 1)^2 (e^{4\pi b\tilde{u}^2} - 1)^2 + (e^{4\pi av_L^2} - 1)^2 + (e^{4\pi b\tilde{u}^2} - 1)^2 \right] \end{aligned}$$

and,

$$\|(e^{4\pi(v_L + \tilde{u})^2(t)} - 1)\|_{L^2(\Omega)}^2 \leq 3(I_1(t) + I_2(t) + I_3(t)),$$

where we set

$$I_1(t) = \int_{\Omega} (e^{4\pi b\tilde{u}^2(t,x)} - 1)^2 dx, \quad I_2(t) = \int_{\Omega} (e^{4\pi av_L^2(t,x)} - 1)^2 dx$$

$$\text{and } I_3(t) = \int_{\Omega} (e^{4\pi av_L^2(t,x)} - 1)^2 (e^{4\pi b\tilde{u}^2(t,x)} - 1)^2 dx.$$

Thanks to the continuity in time of v_L and \tilde{u} and the fact that $\|\nabla u_0\|_{L^2(\Omega)} < 1$, one can choose ε_1 arbitrary small (to be fixed later) and take $0 < A^2 := \frac{1}{2}(1 + \|u_0\|_{H_0^1(\Omega)}^2) < 1$ to find a time $T > 0$ such that for all $t \in [0, T]$

$$\|\tilde{u}(t, \cdot)\|_{H_0^1(\Omega)} \leq \varepsilon_1 \quad \text{and} \quad \|v_L(t, \cdot)\|_{H_0^1(\Omega)} \leq A.$$

Combining Hölder's inequality, the log estimate (1.9) and the fact that for all $t \in [0, T]$,

$$\|v_L(t, \cdot)\|_{H_0^1(\Omega)} \leq A$$

with the monotonicity of the function $x \mapsto x^2 \log(B_1 + \frac{B_2}{x})$ lead to

$$\begin{aligned}
I_2(t) &\leq e^{4\pi a \|v_L(t, \cdot)\|_{L^\infty(\Omega)}^2} \int_{\Omega} (e^{4\pi a v_L^2(t, x)} - 1) dx \\
&\leq \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^\beta \int_{\Omega} (e^{4\pi a v_L^2(t, x)} - 1) dx,
\end{aligned}$$

where we set $\beta = 4\pi a A^2 \lambda$.

Now we choose $\varepsilon > 0$ such that $4\pi a A^2 < 4\pi$. Then, by Moser-Trudinger inequality (1.6)

$$\int_{\Omega} (e^{4\pi a v_L^2(t, x)} - 1) dx \leq C(\Omega, A)$$

and therefore

$$I_2(t) \leq C(\Omega, A) \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^\beta.$$

Now, using (4.30) we have

$$I_1(t) \leq \int_{\Omega} e^{8\pi b \tilde{u}^2(t, x)} - 1 dx.$$

Choosing $\varepsilon_1 > 0$ such that $2b\varepsilon_1^2 \leq 1$, then again by Moser-Trudinger inequality (1.6), we have

$$I_1(t) \leq C(\Omega).$$

Now applying Hölder inequality and (4.30), we obtain

$$\begin{aligned}
I_3(t) &\leq \left(\int_{\Omega} (e^{4\pi a v_L^2(t, x)} - 1)^{2a} dx \right)^{\frac{1}{a}} \left(\int_{\Omega} (e^{4\pi b \tilde{u}^2(t, x)} - 1)^{2b} dx \right)^{\frac{1}{b}} \\
&\leq e^{4\pi a \|v_L(t, \cdot)\|_{L^\infty(\Omega)}^2} \left(\int_{\Omega} (e^{4\pi a^2 v_L^2(t, x)} - 1) dx \right)^{\frac{1}{a}} \left(\int_{\Omega} (e^{4\pi 2b^2 \tilde{u}^2(t, x)} - 1) dx \right)^{\frac{1}{b}},
\end{aligned}$$

and similarly as before, we estimate

$$I_3(t) \leq C \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^\beta.$$

Consequently,

$$\|e^{4\pi(v_L + \tilde{u})^2} - 1\|_{L^2(\Omega)} \leq C \left[1 + \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^{\beta'} \right]$$

where $\beta' = 2\pi a \lambda A^2$. Therefore,

$$\begin{aligned}
\|F(u)\|_{L_T^1(L^2(\Omega))} &\leq C \int_0^T \left(1 + \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^{\beta'} \right) dt \\
&= C \left[T + \int_0^T \left(C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right)^{\beta'} dt \right].
\end{aligned}$$

Choosing λ such that $\beta' < 8$ and applying Hölder inequality with $p = 8/\beta'$, we obtain

$$\begin{aligned}
\|F(u)\|_{L_T^1(L^2(\Omega))} &\lesssim \left[T + T^{1-\frac{\beta'}{8}} \left\| C_\lambda + \frac{\|v_L(t, \cdot)\|_{C^{1/8}}}{A} \right\|_{L_T^8}^{\beta'} \right] \\
&\lesssim \left[T + T^{1-\frac{\beta'}{8}} \left(T^{\frac{\beta'}{8}} + \left(\frac{\|v_L(t)\|_{L_T^8(C^{1/8}(\Omega))}}{A} \right)^{\beta'} \right) \right] \\
&\lesssim \left[T + T^{1-\frac{\beta'}{8}} \left(\frac{\|v_L(t)\|_{L_T^8(C^{1/8}(\Omega))}}{A} \right)^{\beta'} \right] := C(T).
\end{aligned}$$

■

Now we prove the local existence result.

Proof of Theorem 1.4. The proof is divided into two steps

Step 1: The existence in \mathcal{E}_T .

We write the solution u of problem (1.4) as

$$u = v + v_L$$

with as before v_L solves the free wave equation with the same initial data (u_0, u_1) and v solves the following perturbed problem

$$(4.31) \quad \begin{cases} \square v = -(v + v_L)(e^{4\pi(v+v_L)^2} - 1) \\ v(0, x) = 0 \\ \partial_t v(0, x) = 0 \\ v|_{[0, T] \times \partial\Omega} = 0. \end{cases}$$

Define the map $\phi : \mathcal{E}_T \longrightarrow \mathcal{E}_T$; $v \longmapsto \tilde{v}$, where \tilde{v} satisfies

$$(4.32) \quad \begin{cases} \square \tilde{v} = -(v + v_L)(e^{4\pi(v+v_L)^2} - 1) \\ \tilde{v}(0, x) = 0 \\ \partial_t \tilde{v}(0, x) = 0 \\ \tilde{v}|_{[0, T] \times \partial\Omega} = 0. \end{cases}$$

We claim that for T small enough, the map ϕ is well defined from \mathcal{E}_T into itself and is a contraction. Indeed, consider v_1 and v_2 in \mathcal{E}_T and set

$$u_1 = v_1 + v_L \quad u_2 = v_2 + v_L.$$

Using the energy and Strichartz estimates we have

$$\|\phi(v_1) - \phi(v_2)\|_T \leq C \|f(u_1) - f(u_2)\|_{L_T^1(L^2(\Omega))}.$$

Since u_1 and u_2 are two elements of \mathcal{E}_T satisfying $u_1(0, x) = u_2(0, x) = u_0(x)$ and $\|u_0\|_{H_0^1} < 1$, then there exist $0 < A < 1$ and a positive real number T_0 such that for any $0 \leq t \leq T_0$,

$$\|u_1(t)\|_{H_0^1(\Omega)} \leq A \quad \text{and} \quad \|u_2(t)\|_{H_0^1(\Omega)} \leq A.$$

Thanks to Lemma 4.1, there exist $0 < \gamma < 8$ such that for any $T \in [0, T_0]$

$$\begin{aligned}
\|f(u_1) - f(u_2)\|_{L_T^1(L^2(\Omega))} &\leq C\|u_1 - u_2\|_T \left(T + T^{\frac{8-\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right) \\
&\leq C\|v_1 - v_2\|_T \left[T + T^{\frac{8-\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right].
\end{aligned}$$

Let $C(T) = C \left[T + T^{\frac{8-\gamma}{8}} \left(\left(\frac{\|u_1\|_T}{A} \right)^\gamma + \left(\frac{\|u_2\|_T}{A} \right)^\gamma \right) \right]$, we have

$$\|\phi(v_1) - \phi(v_2)\|_T \leq C(T)\|v_1 - v_2\|_T.$$

So, for T small enough, we have $C(T) \leq 1/2$ implying that ϕ is a contraction map. Taking $v_2 = 0$ shows that ϕ is well defined.

Step 2: Uniqueness in the energy space.

Let U_1 and U_2 be in $C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ two solutions of problem (1.4) having the same initial data (u_0, u_1) . Let $w = U_1 - U_2$, then w satisfies

$$\begin{cases} \square w = U_2(e^{4\pi U_2^2} - 1) - U_1(e^{4\pi U_1^2} - 1) \\ w(0, x) = 0 \\ \partial_t w(0, x) = 0 \\ w|_{[0, T] \times \partial\Omega} = 0. \end{cases}$$

In the sequel we shall prove the existence of a continuous function $C(\cdot)$ defined on $[0, T]$, vanishing at $t = 0$ and such that

$$\|w\|_E \leq C(T)\|w\|_E,$$

where $\|w\|_E = \sup_{t \in [0, T]} (\|w(t, \cdot)\|_{H_0^1(\Omega)} + \|\partial_t w(t, \cdot)\|_{L^2(\Omega)})$.

Using the energy estimate, the mean value Theorem and (4.26), we have

$$\begin{aligned}
\|w\|_E &\leq C\|U_2(e^{4\pi U_2^2} - 1) - U_1(e^{4\pi U_1^2} - 1)\|_{L_T^1(L^2(\Omega))} \\
&\leq C\|U_2(e^{4\pi U_2^2} - 1) - (U_2 + w)(e^{4\pi(U_2 + w)^2} - 1)\|_{L_T^1(L^2(\Omega))} \\
&\leq C\|w(e^{4\pi(1+\varepsilon)\overline{w}^2} - 1)\|_{L_T^1(L^2(\Omega))},
\end{aligned}$$

where, $\overline{w} = (1 - \theta)(w + U_2) + \theta U_2 = (1 - \theta)w + U_2$. Thanks to Hölder inequality, the Sobolev embeddings and (4.30), we have

$$\begin{aligned}
\|w(t)(e^{4\pi(1+\varepsilon)\overline{w}^2(t)} - 1)\|_{L^2(\Omega)}^2 &\leq \|w(t)\|_{L^{2+2/\varepsilon}(\Omega)}^2 \|e^{4\pi(1+\varepsilon)\overline{w}^2(t)} - 1\|_{L^{1+\varepsilon}(\Omega)}^2 \\
&\leq \|w(t)\|_{H_0^1(\Omega)}^2 \|e^{4\pi(1+\varepsilon)\overline{w}^2(t)} - 1\|_{L^2(\Omega)}^{2/(1+\varepsilon)}.
\end{aligned}$$

By continuity in time of w and U_2 and the fact that $w(0, x) = \partial_t w(0, x) = 0$ and $U_2(0, x) = u_0(x)$ with $\|\nabla u_0\|_{L^2(\Omega)} < 1$, there exist a positive real number T_1 such that, for any $t \in [0, T_1]$

$$\|w(t)\|_{H_0^1(\Omega)} \leq \varepsilon \quad \text{and} \quad \|U_2(t)\|_{H_0^1(\Omega)} \leq A.$$

Arguing as in the proof of Lemma 4.2, we obtain

$$\begin{aligned} \|e^{4\pi(1+\varepsilon)^2\overline{w}^2(t,\cdot)} - 1\|_{L^2} &\lesssim (1 + \|e^{4\pi(1+\varepsilon)^2aU_2^2(t,\cdot)} - 1\|_{L^2(\Omega)} \\ &+ \|e^{4\pi(1+\varepsilon)^2a^2U_2^2(t,\cdot)} - 1\|_{L^2(\Omega)}). \end{aligned}$$

Finally, for any $0 < T \leq T_1$

$$\begin{aligned} \int_0^T \|(e^{4\pi(1+\varepsilon)^2\overline{w}^2(t)} - 1)\|_{L^2}^{\frac{1}{1+\varepsilon}} dt &\lesssim T + \int_0^T \|(e^{4\pi(1+\varepsilon)^2aU_2^2} - 1)\|_{L^2(\Omega)}^{\frac{1}{1+\varepsilon}} dt \\ &+ \int_0^T \|(e^{4\pi(1+\varepsilon)^2a^2U_2^2} - 1)\|_{L^2(\Omega)}^{\frac{1}{1+\varepsilon}} dt. \end{aligned}$$

To estimate the last two terms in the above right-hand side, we use Lemma 4.2. Hence

$$\int_0^T \|(e^{4\pi(1+\varepsilon)^2\overline{w}^2(t)} - 1)\|_{L^2}^{\frac{1}{1+\varepsilon}} dt \leq C(T)$$

and finally we have

$$\begin{aligned} \|w(e^{4\pi(1+\varepsilon)^2\overline{w}^2} - 1)\|_{L_T^1(L^2(\Omega))} &\leq \sup_{t \in [0, T]} \|w(t)\|_{H_0^1(\Omega)} \int_0^T \|(e^{4\pi(1+\varepsilon)^2\overline{w}^2(t)} - 1)\|_{L^2}^{\frac{1}{1+\varepsilon}} dt \\ &\leq C(T) \|w\|_E \end{aligned}$$

as desired. ■

5. THE GLOBAL EXISTENCE

Theorem 1.4 guarantees that in the subcritical and critical cases, there exists a unique local solution to the Cauchy problem (1.4). In this section we propose to extend the local existence result to global one (in time). We start by the subcritical case and prove Theorem 1.6.

5.1. The subcritical case: Proof of Theorem 1.6.

Proof. We have $E_0 < 1$, so in particular $\|\nabla u_0\|_{L^2(\Omega)} < 1$. Then, according to the local theory (Theorem 1.4), there exist a unique maximal solution u in the space \mathcal{E}_T^* where $0 < T^* \leq +\infty$ is the lifespan of u . The fact that $T^* = +\infty$ is then an immediate consequence of the energy conservation

$$\sup_{0 < t < T^*} \|\nabla u(t, \cdot)\|_{L^2(\Omega)} \leq E(u, t) = E_0 < 1,$$

and the fact that T^* depends upon $1 - \|\nabla u_0\|_{L^2}^2$. ■

The proof in the critical case is more subtle. Indeed, we need to show that concentration cannot occur close to T^* . We combine ideas from [14] and [5]. However, it is important to point out here that our proof is simpler than that one of Burq-Lebeau-Planchon in [5] for the quintic energy critical equation in dimension three. This is because for our purpose, we only use the multipliers u and $\partial_t u$. The multiplier $x \cdot \nabla u$ requires more careful study since it generates other boundary terms but it is not needed here. See [5] for complete details.

5.2. The critical case : Proof of Theorem 1.7.

Proof. Let u be the unique maximal solution to the Cauchy problem (1.4) in the space \mathcal{E}_T^* . We show that if T^* is finite then we have a contradiction. We start by showing some properties of the maximal solution u in the critical case.

Proposition 5.1. *The maximal solution u verifies*

$$(5.33) \quad \limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\Omega)} = 1,$$

and

$$(5.34) \quad u(t) \xrightarrow{t \rightarrow T^*} 0 \quad \text{in } L^2(\Omega).$$

Proof. Using (1.5), we have for all $0 \leq t < T^*$,

$$\|\nabla u(t)\|_{L^2(\Omega)}^2 + \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{e^{4\pi u^2(t,x)} - 1 - 4\pi u^2(t,x)}{4\pi} dx = 1.$$

Hence ,

$$\limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\Omega)} \leq 1.$$

Assuming that $\limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\Omega)} = \ell < 1$, then for $\varepsilon := \frac{1-\ell}{2}$, one can find a time t_0 such that for all $0 < t_0 < t < T^*$, we have

$$\|\nabla u(t)\|_{L^2(\Omega)} \leq \frac{\ell + 1}{2}.$$

Moreover, by continuity, there exists a time t_1 in the interval $[0, t_0]$, such that

$$\sup_{0 \leq t \leq t_0} \|\nabla u(t)\|_{L^2(\Omega)} = \|\nabla u(t_1)\|_{L^2(\Omega)} < 1.$$

Hence

$$\sup_{0 \leq t < T^*} \|\nabla u(t)\|_{L^2(\Omega)} < 1.$$

Consequently, u can be extended beyond the time T^* , a contradiction.

Now, let us show (5.34). We consider a sequence (t_n) converging to T^* as $n \rightarrow +\infty$. We start by proving that $u_n := u(t_n)$ is a Cauchy sequence in $L^2(\Omega)$. Indeed,

$$\begin{aligned} \|u(t_n) - u(t_m)\|_{L^2(\Omega)} &\leq |t_n - t_m| \sup_{\tau \in [0, T^*)} \|\partial_t u(\tau)\|_{L^2(\Omega)} \\ &< |t_n - t_m|, \end{aligned}$$

which can be made arbitrary small. Thus, there exists \bar{u} in $L^2(\Omega)$ such that $u(t)$ converges to \bar{u} in $L^2(\Omega)$ as $t \rightarrow T^*$. Now, we prove that $\bar{u} = 0$. Using (1.5) and Fatou's Lemma, we have

$$\begin{aligned} \limsup_{t \rightarrow T^*} \|\nabla u(t)\|_{L^2(\Omega)}^2 - 1 &\leq - \liminf_{t \rightarrow T^*} \|\partial_t u(t)\|_{L^2(\Omega)}^2 - \liminf_{t \rightarrow T^*} \int_{\Omega} \frac{e^{4\pi u^2(t,x)} - 1 - 4\pi u^2(t,x)}{4\pi} dx \\ &\leq - \liminf_{t \rightarrow T^*} \|\partial_t u(t)\|_{L^2(\Omega)}^2 - \int_{\Omega} \liminf_{t \rightarrow T^*} \frac{e^{4\pi u^2(t,x)} - 1 - 4\pi u^2(t,x)}{4\pi} dx. \end{aligned}$$

By (5.33)

$$\liminf_{t \rightarrow T^*} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \liminf_{t \rightarrow T^*} \frac{e^{4\pi u^2(t,x)} - 1 - 4\pi u^2(t,x)}{4\pi} dx \leq 0,$$

which implies

$$\liminf_{t \rightarrow T^*} (e^{4\pi u^2(t,x)} - 1 - 4\pi u^2(t,x)) = 0$$

Therefore, $\bar{u} = 0$. This completes the proof of Proposition 5.1.

■

Now we construct a sort of “critical element” in the sense that all its energy concentrates in the backward light cone issued from a point. Since the equation is invariant under time translation, in the sequel we will assume that $T^* = 0$.

Proposition 5.2. *Let u be the maximal solution of problem (1.4). Then, there exists a point x^* in $\bar{\Omega}$ such that, for all $t < 0$*

$$(5.35) \quad \text{supp } \nabla u(t, \cdot) \subset B(x^*, -t) \cap \bar{\Omega}, \quad \text{supp } \partial_t u(t, \cdot) \subset B(x^*, -t) \cap \bar{\Omega}.$$

The proof goes along the same lines as in [14]. For the convenience of the reader, we sketch it here.

Proof. **Claim 1:** There exists a point x^* in $\bar{\Omega}$ such that for all $r > 0$, we have

$$(5.36) \quad \limsup_{t \rightarrow 0^-} \int_{\{x; |x-x^*| \leq r\} \cap \bar{\Omega}} |\nabla u(t)|^2 dx = 1.$$

Indeed, by contradiction and as in [14], there exist two positive real numbers r and η such that for any $x \in \bar{\Omega}$ we have

$$(5.37) \quad \limsup_{t \rightarrow 0^-} \int_{\{y; |x-y| \leq r\} \cap \bar{\Omega}} e(u)(t, y) dy \leq 1 - \eta.$$

Now let $x \in \bar{\Omega}$ and define the cut-off function φ_x by $0 \leq \varphi_x \leq 1$, $\varphi_x \equiv 1$ in $B(x, r/2) \cap \bar{\Omega}$ and $\varphi_x \equiv 0$ outside $B(x, r) \cap \bar{\Omega}$. Obviously, from (5.37) and Proposition 5.1, we have

$$\limsup_{t \rightarrow 0^-} \int_{\{y; |x-y| \leq r\} \cap \bar{\Omega}} e(\varphi_x u, \varphi_x \partial_t u)(t) dy \leq 1 - \eta.$$

Now choose a time $t_1 > T^* - r/8$ such that

$$\int_{\{y; |x-y| \leq r\} \cap \bar{\Omega}} e(\varphi_x u, \varphi_x \partial_t u)(t_1) dy \leq 1 - \eta/2.$$

From the local theory (Theorem 1.4), one can solve globally in time problem (1.4) with the initial data $(\varphi_x u(t_1, \cdot), \varphi_x \partial_t u(t_1, \cdot))$. By the finite speed of propagation, we deduce that u can be continued in the backward light cone of vertex $(x, t_1 + r/2)$. Since the set $\bar{\Omega}$ is compact, then we can extract a finite covering from $\bar{\Omega} = \cup_{x \in \bar{\Omega}} B(x, r) \cap \bar{\Omega}$. This implies that u can be continued beyond its lifetime T^* which is a contradiction.

Claim 2: We have the following

$$(5.38) \quad \lim_{t \rightarrow 0^-} \int_{\{x; |x-x^*| \leq -t\} \cap \bar{\Omega}} |\nabla u(t)|^2 dx = 1.$$

$$(5.39) \quad \forall t < 0, \quad \int_{\{x, |x-x^*| \leq -t\} \cap \bar{\Omega}} e(u(t)) dx = 1.$$

Indeed, without loss of generality, we can assume that $x^* = 0$. The proof of (5.38) is straightforward. Suppose that (5.38) is false. Then, there exists a sequence of negative real number (t_n) tending to zero such that

$$\forall n \in \mathbb{N}, \quad \int_{|x| \leq -t_n} |\nabla u(t_n)|^2 dx \leq 1 - \eta \quad \text{for some } 0 < \eta < 1.$$

Then, arguing as in the proof of the previous claim, the solution can be continued beyond T^* , a contradiction. To prove (5.39), fix $\varepsilon > 0$. By (5.38), there exists a time $t_\varepsilon < 0$ such that $\int_{|x| \leq -t} |\nabla u(t)|^2 dx \geq 1 - \varepsilon$ for $t_\varepsilon \leq t < 0$. By the finite speed of propagation, we deduce that

$$\forall t < 0, \quad \int_{|x| \leq -t} e(u)(t) dx \geq 1 - \varepsilon.$$

Letting ε go to zero, we obtain the desired result.

Now, the proof of Proposition 5.2 is immediate. If for a fixed $t < 0$, the support property is not satisfied, then there exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for all $x_0 \in \bar{\Omega}$, we have

$$\int_{\{x, |x-x_0| \geq (1+\eta_0)(-t)\} \cap \bar{\Omega}} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 dx \geq \varepsilon_0.$$

But for $x_0 = x^*$, the above inequality together with (5.39) contradict the fact that the $E(u, t) = 1$.

■

Proof of Theorem 1.7.

Multiplying equation (1.4) by $2\partial_t u$, we obtain

$$(5.40) \quad \partial_t(e(u)) - \operatorname{div}_x(2\partial_t u \cdot \nabla u) = 0,$$

where the energy density $e(u)$ is defined by (2.14).

Integrating (5.40) over the backward truncated cone K_S^T ($S < T < 0$), we get

$$(5.41) \quad \int_{K_S^T} \operatorname{div}_{t,x} \vec{B}(t, x) dx dt = 0,$$

where

$$\vec{B} = (B_0, B_1, B_2), \quad B_0 = e(u) \quad \text{and} \quad B_j = -2\partial_t u \frac{\partial u}{\partial x_j}, \quad j = 1, 2.$$

Thanks to Stokes formula, we obtain

$$\begin{aligned} \int_{D(T)} e(u)(T) dx - \int_{D(S)} e(u)(S) dx & - \int_{\{([S, T] \times \partial\Omega)\} \cap K_S^T} \nu(x) \cdot (2\partial_t u \nabla u) d\sigma \\ & + \frac{1}{\sqrt{2}} \int_{M_S^T} \left\{ |\partial_t u \frac{x}{|x|} + \nabla u|^2 + \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{4\pi} \right\} d\sigma = 0, \end{aligned}$$

here M_S^T defined by (2.13) and $\nu(x)$ is the exterior normal vector to Ω at point x . Taking into account the Dirichlet boundary condition, we have

$$(5.42) \quad \int_{D(S)} e(u(S)) dx - \int_{D(T)} e(u(T)) dx = \int_{M_S^T} \left\{ |\partial_t u \frac{x}{|x|} + \nabla u|^2 + \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{4\pi} \right\} \frac{d\sigma}{\sqrt{2}}.$$

Now, multiplying equation (1.4) by $2u$, integrating over the backward truncated cone K_S^T and using Stokes formula given the Dirichlet condition, we obtain

$$(5.43) \quad \int_{D(T)} \partial_t u(T) u(T) \, dx - \int_{D(S)} \partial_t u(S) u(S) \, dx \\ + \frac{1}{\sqrt{2}} \int_{M_S^T} (\partial_t u + \nabla u \cdot \frac{x}{|x|}) u \, d\sigma + \int_{K_S^T} (|\nabla u|^2 - |\partial_t u|^2 + u^2(e^{4\pi u^2} - 1)) \, dx \, dt = 0.$$

Thanks to (5.35), identity (5.42) implies that

$$(5.44) \quad \frac{1}{\sqrt{2}} \int_{M_S^T} \{ |\partial_t u \frac{x}{|x|} + \nabla u|^2 + \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{4\pi} \} \, d\sigma = 0.$$

Since $u(t) \rightarrow 0$ in $L^2(\Omega)$ and $\|\nabla u(t)\|_{L^2(\Omega)} \rightarrow 1$ as t goes to 0, the energy identity (1.5) implies that

$$(5.45) \quad \partial_t u(t) \rightarrow 0 \text{ in } L^2(\Omega).$$

Letting T go to zero in (5.43), using (5.45) and (5.44), we have

$$- \int_{D(S)} \partial_t u(S) u(S) \, dx + \int_{K_S^0} (|\nabla u|^2 - |\partial_t u|^2 + u^2(e^{4\pi u^2} - 1)) \, dx \, dt = 0.$$

Multiplying the above identities by $\frac{-1}{S}$, we deduce that

$$\int_{D(S)} \partial_t u(S) \frac{u(S)}{S} \, dx \leq \frac{1}{S} \int_{K_S^0} |\nabla u|^2 \, dx \, dt - \frac{1}{S} \int_{K_S^0} |\partial_t u|^2 \, dx \, dt.$$

Thanks to the mean value Theorem, there exist $t_0 \in]S, 0[$ such that

$$\frac{1}{S} \int_{K_S^0} |\nabla u|^2 \, dx \, dt = - \int_{|x-x^*| \leq -t_0} |\nabla u(t_0, x)|^2 \, dx.$$

So, using (5.33)

$$\frac{1}{S} \int_{K_S^0} |\nabla u|^2 \, dx \, dt \xrightarrow{S \rightarrow 0^-} -1.$$

Similarly

$$\frac{1}{S} \int_{K_S^0} |\partial_t u|^2 \, dx \, dt \xrightarrow{S \rightarrow 0^-} 0.$$

Moreover, since $\left| \frac{u(S)}{S} \right| = \left| \frac{1}{S} \int_0^S \partial_t u(\tau) \, d\tau \right|$, then $(\frac{u(S)}{S})$ is bounded in $L^2(\Omega)$. Hölder inequality combined to the above result imply

$$\int_{D(S)} \partial_t u(S) \frac{u(S)}{S} \, dx \xrightarrow{S \rightarrow 0^-} 0,$$

leading to $0 \leq -1$, a contradiction. ■

6. ILL-POSEDNESS IN THE SUPERCRITICAL CASE

In this section we prove the instability result given by Theorem 1.8. The construction is similar to that one in Proposition 3.2. However here, we have to consider the nonlinear problem and not just the linear one. In particular, we will show that the solution to the ODE (the nonlinear wave equation without the diffusion term) is a “perturbation” of the cosine function. We construct a slightly supercritical initial data given through the same functions f_k as in (3.25). The concentration presented in the data yields fast periodic oscillations in the ODE regime. Moreover, the special form of the data and the finite speed of propagation allow us to conclude that solutions of the P.D.E. and the ODE coincide in a backward light cone.

• **Step 1: Construction of the initial data.**

Without loss of generality, we can assume that $0 \in \Omega$. Choose $0 < \eta < 1$ small enough such that the ball $B(0, \eta) \subset \Omega$. For $k \geq 1$, let v_k solve

$$\square v_k + v_k(e^{4\pi v_k^2} - 1) = 0, \quad v_k(0, x) = (1 + \frac{1}{k})f_k(\frac{x}{\eta}), \quad \partial_t v_k(0, x) = 0, \quad v_k|_{\partial\Omega} = 0$$

and w_k the solution of

$$\square w_k + w_k(e^{4\pi w_k^2} - 1) = 0, \quad w_k(0, x) = f_k(\frac{x}{\eta}), \quad \partial_t w_k(0, x) = 0, \quad w_k|_{\partial\Omega} = 0.$$

Since,

$$(6.46) \quad \|\nabla f_k(\frac{\cdot}{\eta})\|_{L^2(\Omega)}^2 = \int_{\eta e^{-k/2} \leq |x| \leq \eta} \frac{1}{k\pi|x|^2} dx = \frac{2}{k} \int_{\eta e^{-k/2}}^{\eta} \frac{dr}{r} = 1,$$

we easily verify that given $\varepsilon > 0$, then using Poincaré inequality

$$\|v_k(0) - w_k(0)\|_{H_0^1(\Omega)}^2 + \|\partial_t v_k(0) - \partial_t w_k(0)\|_{L^2(\Omega)}^2 = \frac{1}{k^2} \|f_k(\frac{\cdot}{\eta})\|_{H_0^1(\Omega)}^2 \leq \frac{C}{k^2} \leq \varepsilon,$$

for k large enough. Therefore w_k and v_k satisfy (1.10). Now, we will show that the initial data associated to v_k and w_k are slightly supercritical.

$$E(w_k, 0) = \|\nabla f_k(\frac{\cdot}{\eta})\|_{L^2(\Omega)}^2 + \frac{1}{4\pi} \int_{\Omega} e^{4\pi f_k^2(\frac{\cdot}{\eta})} - 1 - 4\pi f_k^2(\frac{\cdot}{\eta}) dx \leq 1 + \frac{1}{4\pi} \int_{\Omega} (e^{4\pi f_k^2(\frac{\cdot}{\eta})} - 1) dx.$$

But,

$$\begin{aligned} \frac{1}{4\pi} \int_{\Omega} (e^{4\pi f_k^2(\frac{\cdot}{\eta})} - 1) dx &= \frac{1}{4\pi} \left(\int_{\eta e^{-k/2} \leq |x| \leq \eta} (e^{\frac{4}{k} \log^2(\frac{|x|}{\eta})} - 1) dx + \int_{|x| \leq \eta e^{-k/2}} (e^k - 1) dx \right) \\ &= \frac{\eta^2}{2} \int_{e^{-k/2}}^1 r (e^{\frac{4}{k} \log^2 r} - 1) dr + \frac{e^k - 1}{2} \int_0^{\eta e^{-k/2}} r dr. \\ &= \frac{\eta^2}{2} \int_{e^{-k/2}}^1 r e^{\frac{4}{k} \log^2 r} dr, \end{aligned}$$

and to estimate the last integral, we use the following Lemma (see [16]).

Lemma 6.1. *For any $a \geq 1$ and $k \in \mathbb{N}$,*

$$I(a, k) := \int_{e^{-k/2}}^1 r e^{\frac{4a^2}{k} \log^2 r} dr \leq 2e^{(a^2-1)k}.$$

Applying the above Lemma with $a = 1$, we get

$$\frac{\eta^2}{2} \int_{e^{-k/2}}^1 r e^{\frac{4}{k} \log^2 r} dr \leq \eta^2.$$

Hence, for k large enough, $E(w_k, 0) \leq 1 + \eta^2$.

Similarly, we prove that $0 < E(v_k, 0) - 1 \leq C\eta^2 e^{2+\frac{1}{k}}$. Therefore, for k large enough

$$0 < E(v_k, 0) - 1 \leq 3\eta^2 e^3.$$

• **Step 2: Approximation.**

Let ϕ_k and ψ_k be the two solutions of the following ordinary differential equation (O.D.E.)

$$(6.47) \quad \ddot{y} + y(e^{4\pi y^2} - 1) = 0,$$

with initial data

$$\phi_k(0) = (1 + \frac{1}{k})\sqrt{\frac{k}{4\pi}} \quad , \quad \dot{\phi}_k(0) = 0,$$

and

$$\psi_k(0) = \sqrt{\frac{k}{4\pi}} \quad , \quad \dot{\psi}_k(0) = 0.$$

Since $v_k = \phi_k$ and $w_k = \psi_k$ on the ball $B = \{(x, t = 0) : |x| \leq \eta e^{-k/2}\}$ in the hyperplane $t = 0$, then by finite speed of propagation $v_k = \phi_k$ and $w_k = \psi_k$ in the backward light cone

$$K = \{(x, t) / t = \alpha \eta e^{-k/2} \quad |x| \leq (1 - \alpha) \eta e^{-k/2} ; 0 \leq \alpha \leq 1\}.$$

• **Step 3: Decoherence.**

We start by recalling the following result (for example, see section III.5 from [2]).

Lemma 6.2. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a regular function and consider the following O.D.E.*

$$(6.48) \quad \ddot{Y}(t) + F'(Y(t)) = 0 \quad , \quad (Y(0), \dot{Y}(0)) = (Y_0, 0)$$

where $Y_0 > 0$. Then equation (6.48) has a periodic non constant solution if and only if the function $G : z \mapsto 2(F(Y_0) - F(z))$ has two simples distinct zeros α and β with $\alpha \leq Y_0 \leq \beta$ and such that G has no zero in the interval $]\alpha, \beta[$. In this case, the period is given by

$$(6.49) \quad T = 2 \int_{\alpha}^{\beta} \frac{dz}{\sqrt{G(z)}} = 2 \int_{\alpha}^{\beta} \frac{dz}{\sqrt{2(F(Y_0) - F(z))}}.$$

Taking $F(z) = \frac{e^{4\pi z^2} - 1 - 4\pi z^2}{8\pi}$ in the above Lemma, the solution ϕ_k is periodic and we have

$$\begin{aligned} T_k &= 2 \int_{-(1+\frac{1}{k})\sqrt{\frac{k}{4\pi}}}^{(1+\frac{1}{k})\sqrt{\frac{k}{4\pi}}} \frac{dz}{\sqrt{2[(e^{(1+\frac{1}{k})^2 k} - (1+\frac{1}{k})^2 k) - (e^{4\pi z^2} - 4\pi z^2)]}} \\ &= 4 \int_0^{(1+\frac{1}{k})\sqrt{k}} \frac{du}{\sqrt{(e^{(1+\frac{1}{k})^2 k} - (1+\frac{1}{k})^2 k) - (e^{u^2} - u^2)}}. \end{aligned}$$

Now to estimate the period T_k we use the following Lemma.

Lemma 6.3. *For any $A > 1$, we have*

$$\int_0^A \frac{du}{\sqrt{(e^{A^2} - A^2) - (e^{u^2} - u^2)}} \leq \sqrt{1 - 2e^{-1}} e^{\frac{-A^2}{2}} \left[A - \frac{1}{A} + \frac{A}{A^2 - 1} \right].$$

Proof. Write,

$$\int_0^A \frac{du}{\sqrt{(e^{A^2} - A^2) - (e^{u^2} - u^2)}} = \int_0^{A - \frac{1}{A}} + \int_{A - \frac{1}{A}}^A$$

The first term on the right-hand side can be estimated by $\sqrt{1 - 2e^{-1}} (A - \frac{1}{A}) e^{\frac{-A^2}{2}}$. Let $h(u) = \frac{1}{u(e^{u^2} - 1)}$ and $g'(u) = \frac{u(e^{u^2} - 1)}{\sqrt{(e^{A^2} - A^2) - (e^{u^2} - u^2)}}$. Integrating by parts in the second integral, we obtain

$$(6.50) \quad \int_{A - \frac{1}{A}}^A \frac{du}{\sqrt{(e^{A^2} - A^2) - (e^{u^2} - u^2)}} \leq \frac{A}{A^2 - 1} \sqrt{e^{A^2} - e^{A^2 - 2 - \frac{1}{A^2}} - 2 + \frac{1}{A^2}} \\ \leq \sqrt{1 - 2e^{-1}} \frac{A}{A^2 - 1} e^{\frac{-A^2}{2}},$$

giving,

$$\int_0^A \frac{du}{\sqrt{(e^{A^2} - A^2) - (e^{u^2} - u^2)}} \leq \sqrt{1 - 2e^{-1}} e^{\frac{-A^2}{2}} \left[A - \frac{1}{A} + \frac{A}{A^2 - 1} \right]$$

as desired. ■

Choosing $A = \sqrt{k}(1 + \frac{1}{k})$ in the above Lemma 6.3 with k large enough, we get

$$T_k \leq \sqrt{k} e^{\frac{-k}{2}(1 + \frac{1}{k})^2} 4e^2 \left[\frac{(k+1)^2 - k}{k(k+1)} + \frac{(k+1)}{(k+1)^2 - k} \right] \\ \leq C_1 \sqrt{k} e^{\frac{-k}{2}(1 + \frac{1}{k})^2}.$$

Since ϕ_k is a periodic function and decreasing on $]0, T_k/4[$ (actually, ϕ_k may be viewed as a cosine function) then, we choose $t_k \in]0, T_k/4[$ such that

$$\phi_k(t_k) = (1 + \frac{1}{k}) \sqrt{\frac{k}{4\pi}} - \left((1 + \frac{1}{k}) \sqrt{\frac{k}{4\pi}} \right)^{-1}.$$

Clearly,

$$t_k = \int_{\sqrt{k} + \frac{1}{\sqrt{k}} - \frac{4\pi\sqrt{k}}{k+1}}^{\sqrt{k} + \frac{1}{\sqrt{k}}} \frac{du}{\sqrt{(e^{(1 + \frac{1}{k})^2 k} - (1 + \frac{1}{k})^2 k) - (e^{u^2} - u^2)}}.$$

Using (6.50) with $A = \sqrt{k} + \frac{1}{\sqrt{k}}$, we obtain

$$t_k \leq e^{8\pi} \frac{k(k+1)}{\sqrt{k}(k^2 + (2 - 4\pi)k + 1)} e^{-\frac{1}{2}(\sqrt{k} + \frac{1}{\sqrt{k}})^2} \\ \leq e^{8\pi} \frac{e^{-k/2}}{\sqrt{k}} \frac{k(k+1)}{(k^2 + (2 - 4\pi)k + 1)}.$$

Then, if k is large enough

$$t_k \leq \frac{\eta}{2} e^{-k/2}.$$

Finally, we will prove that this time t_k is sufficient to establish the instability result. Since,

$$\begin{aligned} \|\partial_t(v_k - w_k)(t_k)\|_{L^2(\Omega)}^2 &= \int_{\Omega} |\partial_t(v_k - w_k)(t_k)|^2 dx \\ &\geq \int_{|x| < \frac{\eta}{2} e^{-k/2}} |\partial_t(v_k - w_k)(t_k)|^2 dx = \frac{\pi}{4} \eta^2 e^{-k} |\partial_t(\phi_k - \psi_k)(t_k)|^2. \end{aligned}$$

Then, it suffices to estimate $|\partial_t(\phi_k - \psi_k)(t_k)|$. To do so, we can write

$$|\partial_t(\phi_k - \psi_k)(t_k)| = \frac{|(\partial_t \phi_k(t_k))^2 - (\partial_t \psi_k(t_k))^2|}{|\partial_t \phi_k(t_k) + \partial_t \psi_k(t_k)|},$$

with

$$(6.51) \quad \partial_t \phi_k(t_k)^2 = \frac{e^{4\pi \phi_k(0)^2} - 4\pi \phi_k(0)^2 - e^{4\pi \phi_k(t_k)^2} + 4\pi \phi_k(t_k)^2}{4\pi}$$

and similarly

$$(6.52) \quad \partial_t \psi_k(t_k)^2 = \frac{e^{4\pi \psi_k(0)^2} - 4\pi \psi_k(0)^2 - e^{4\pi \psi_k(t_k)^2} + 4\pi \psi_k(t_k)^2}{4\pi}.$$

Hence,

$$|(\partial_t \phi_k(t_k))^2 - (\partial_t \psi_k(t_k))^2| = \left| \frac{e^{4\pi \phi_k(0)^2} - e^{4\pi \phi_k(t_k)^2} - e^{4\pi \psi_k(0)^2} + e^{4\pi \psi_k(t_k)^2}}{4\pi} \right|.$$

Using the fact that ψ_k is decreasing on $[0, T_k/4]$, we have

$$|e^{4\pi \psi_k(0)^2} - 4\pi \psi_k(0)^2 - e^{4\pi \psi_k(t_k)^2} + 4\pi \psi_k(t_k)^2| \leq 2e^k.$$

In addition,

$$e^{4\pi \phi_k(0)^2} - e^{4\pi \phi_k(t_k)^2} = e^{k + \frac{1}{k} + 2} - e^{2 - 8\pi + k + \frac{1}{k} + \frac{16\pi^2 k}{(k+1)^2}}.$$

Therefore for k large enough,

$$|(\partial_t \phi_k(t_k))^2 - (\partial_t \psi_k(t_k))^2| \geq C e^k.$$

Moreover,

$$\begin{aligned} |\partial_t \phi_k(t_k) + \partial_t \psi_k(t_k)| &\leq \frac{e^{2\pi \phi_k(0)^2} + e^{2\pi \psi_k(0)^2}}{\sqrt{4\pi}} \\ &\leq \frac{e^{k/2} + e^{\frac{k}{2}(1 + \frac{1}{k})^2}}{\sqrt{4\pi}} \\ &\leq \frac{e^{k/2}(e^{1 + \frac{1}{k^2} + \frac{2}{k}} + 1)}{\sqrt{4\pi}}. \end{aligned}$$

For large k , we have

$$|\partial_t \phi_k(t_k) + \partial_t \psi_k(t_k)| \leq C e^{k/2},$$

and consequently,

$$|\partial_t(\phi_k - \psi_k)(t_k)|^2 \geq C e^k.$$

Finally, we obtain

$$(6.53) \quad \liminf_{k \rightarrow \infty} \|\partial_t(v_k - w_k)(t_k)\|_{L^2(\Omega)}^2 \geq \frac{\pi}{4} C \eta^2.$$

This finishes the proof of Theorem 1.8. ■

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